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**LETTER TO THE EDITOR**

# Spacing distributions for rhombus billiards

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**Abstract.** We show that the spacing distributions of rational rhombus billiards fall in a family of universality classes distinctly different from the Wigner–Dyson family of random matrix theory and the Poisson distribution. Some of the distributions find explanation in a recent work of Bogomolny, Gerland, and Schmit. For the irrational billiards, despite ergodicity, we get the same distribution for the examples considered—once again, distinct from the Wigner–Dyson distributions. All the results are obtained numerically by a method that allows us to reach very high energies.

Statistical analysis of level correlations of a quantum system is one of the many ways to study the effects of chaotic behaviour of its classical counterpart [1]. For such complex systems, the fluctuations are very well described by the random matrix theory, giving rise to three classes of universality corresponding to orthogonal, unitary and symplectic ensembles (OE, UE and SE). On the other hand, for integrable systems, the short-range correlations follow the Poisson distribution. Rhombus billiards [2] are peculiar as they are pseudo-integrable systems and for this reason their statistical properties belong to another class of universality [3]. These non-integrable systems are termed pseudo-integrable as the dynamics occurs on a multiply-connected, compact surface in the phase space. For example, in the case of $\pi/3$-rhombus billiard, the invariant integral surface is a sphere with two handles [2, 4]. It has been shown that the short-range properties (spacing distribution) can be fitted by Brody distributions [5] with parameters depending on the genus [6]. However, a very small number of levels were used to achieve the statistics and, as it was outlined by the authors, the parameters were smoothly changing with the number of levels considered. This last effect is probably a consequence of the pseudo-integrability and thus one has to consider levels lying very high in energy to have converged statistics. Furthermore, Brody distributions are not very convenient for two reasons: (i) they are not on a firm theoretical basis like random matrix theory and so one cannot gain too much knowledge about the system from the Brody parameter; (ii) their behaviour at small spacing is not linear, whereas it is so for rhombus billiards. In contrast, in a recent paper [7], Bogomolny *et al* have proposed a model derived from the Dyson's stochastic Coulomb gas model [8, 9]: eigenvalues are considered as classical particles on a line, with a two-body interaction potential given by $V(x) = -\ln(x)$. In contrast to Dyson's model, where all possible pairs are considered, the same interaction is restricted only to nearest-neighbours. Hereafter, this model will be referred to as the short-ranged Dyson's model (SRDM). The joint probability obtained

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gives rise to spacing distributions showing linear level repulsion and exponential decrease for large spacing. More precisely, the nearest-neighbour (NN) and next-nearest-neighbour (NNN) distributions are

$$P(s) = 4s\mathrm{e}^{-2s} \qquad \text{and} \qquad P_2(s) = \tfrac{8}{3}s^3\mathrm{e}^{-2s}. \tag{1}$$

It is worth noting that exactly the same functional form was used in the past [10] to explain the intermediate spacing distribution for a rectangle billiard with a flux line—an Aharanov–Bohm billiard. In [7] it is also shown that the level statistics of some rhombus billiards agree very well with these distributions. However, only rhombi with rational angles and with Dirichlet boundary conditions on both the $x$- and $y$-axis (i.e. right-angled triangle) were studied. In this letter, we extend the preceding study to rational billiards with Neumann boundary conditions (i.e. 'pure' rhombus) and also to irrational billiards (both classes of boundary conditions). Of course, in a rhombus, making the shorter (longer) diagonal Neumann means that one is considering a larger obtuse (acute) triangle. So, the modifications are expected but here they are non-trivial.

The spectral properties of these systems which are non-integrable and yet non-chaotic is thus an important unsettled problem. The solution of this problem is partly in devising numerical techniques that allow one to go to higher energies, and, partly in developing statistical models like the SRDM [7] mentioned above. In this letter, we first discuss the method and then use levels in the high-energy range to show agreements and disagreements with the results in [7]. To give an idea, the efficiency of the method is such that we were able to compute a very large number of levels (up to 36 000 for a given rhombus and a given symmetry class), so that the statistical properties are fully converged. In the latter part of this letter, we show the effects of both the boundary conditions and the irrationality on the level spacing distributions.

The Schrödinger equation for a particle moving freely in a rhombus billiard (shown by figure 1) is simply

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi(x,y) = E\psi(x,y) \tag{2}$$

with the additional condition that $\psi(x, y)$ is vanishing on the boundary (Dirichlet conditions). The geometry of the system leads to a natural change of coordinates: the

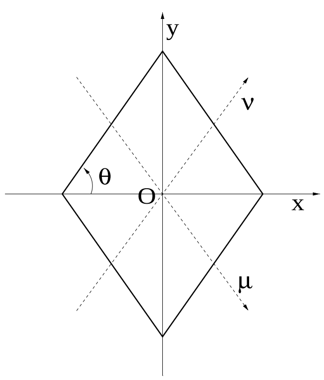


**Figure 1.** Rhombus-shaped enclosure in which the particle moves freely with elastic bounces on the boundary. The quantum problem corresponds to imposing the Dirichlet conditions for the wavefunctions. The system being symmetric under reflections with respect to the $x$-axis or the $y$-axis, Dirichlet or Neumann boundary conditions can be imposed on both the axes, leading to four different classes of symmetry. By considering axes crossing at the centre $O$ of the system and parallel to the edges of the billiard, a non-orthogonal coordinate system $(\mu, \nu)$ is constructed in which the Dirichlet boundary conditions on the enclosure separate (see text).

two new axes cross at the centre and are parallel to the edges of the billiard (see figure 1):

$$\mu = \frac{1}{2}\left(\frac{x}{\cos\theta} - \frac{y}{\sin\theta}\right)$$
$$\nu = \frac{1}{2}\left(\frac{x}{\cos\theta} + \frac{y}{\sin\theta}\right). \tag{3}$$

In this new coordinate system, the original rhombus is mapped onto a square of length $L$ and thus, in this coordinate system, the boundary conditions separate, of course at the price of a slightly more complicated Schrödinger equation:

$$-\frac{\hbar^2(\partial^2_{\mu\mu} + \partial^2_{\nu\nu} - 2\cos(2\theta)\partial^2_{\mu\nu})}{2m\sin^2(2\theta)}\psi(\mu,\nu) = E\psi(\mu,\nu). \tag{4}$$

The change $\mu \to \frac{2}{L}\mu$, $\nu \to \frac{2}{L}\nu$ and $E \to (\frac{2}{L})^2\frac{m}{\hbar^2}E$ gives rise to the scaled Schrödinger equation (after multiplication by $2\sin^2(2\theta)$):

$$-(\partial^2_{\mu\mu} + \partial^2_{\nu\nu} - 2\cos(2\theta)\partial^2_{\mu\nu})\psi = 2\sin^2(2\theta)E\psi \tag{5}$$

the boundary condition being then at the points $\mu = \pm 1$ and $\nu = \pm 1$.

To solve the eigenvalue problem, a possible idea is to expand any wavefunction in a basis satisfying the boundary conditions

$$\psi(\mu,\nu) = \sum_{n_\mu,n_\nu=0}^{\infty} a(n_\mu,n_\nu)\phi_{n_\mu}(\mu)\phi_{n_\nu}(\nu). \tag{6}$$

The simplest choice is the Fourier sine and cosine series. Unfortunately, the operator $\partial^2_{\mu\nu}$ has no selection rules in this basis, thus the matrix representation of the left part of the Schrödinger equation (5) is totally filled. Numerically, we will approximate the wavefunction by keeping only a (large) number of terms in the preceding series. For this system and for many other Coulomb-like systems, it has been observed that the rate of convergence of the series is much slower when the matrix is filled than when selection rules occur.

To avoid this difficulty, we introduce the following basis for each coordinate $\mu$ and $\nu$:

$$\phi_n(u) = (1-u^2)C_n^{(\frac{3}{2})}(u) \tag{7}$$

where $C_n^\alpha$ are Gegenbauer polynomials [11]. This basis is complete and all operators appearing in equation (5) have selection rules. More precisely, we have

$$|\Delta n_\mu|, |\Delta n_\nu| \leqslant 2 \qquad \Delta n_\mu + \Delta n_\nu = 0, \pm 2, \pm 4. \tag{8}$$

Furthermore, all matrix elements are analytically known and are given by simple polynomial expressions of the two quantum numbers $(n_\mu, n_\nu)$. The only difficulty is the non-orthogonality of the basis: that is $\langle n'|n\rangle$ does not reduce to $\delta_{nn'}$ but also shows the preceding selection rules.

This basis also allows us to take directly into account the symmetries of the original problem, namely the reflections with respect to the $x$-axis ($S_x$) or the $y$-axis ($S_y$). In $(\mu,\nu)$ coordinates, they become

$$S_x \begin{cases} \mu \to \nu \\ \nu \to \mu \end{cases} \qquad S_y \begin{cases} \mu \to -\nu \\ \nu \to -\mu. \end{cases} \tag{9}$$

Using the properties of the Gegenbauer polynomials [11] we are able to construct four different bases in which the two operators $S_x$ and $S_y$ are simultaneously diagonal with eigenvalues, $\epsilon_x = \pm 1$ and $\epsilon_y = \pm 1$. Of course, this transformation preserves the selection rules and hence the band structure. We shall denote the eigenfunctions vanishing on both the diagonals by $(--)$ and not vanishing on either by $(++)$ parity classes.

The original Schrödinger equation is thus transformed to a generalized eigenvalue problem:

$$A|\psi\rangle = EB|\psi\rangle \tag{10}$$

where $A$ and $B$ are real, sparse and banded matrices. This kind of system is easily solved using the Lanczos algorithm [12]. It is an iterative method, highly efficient to obtain few eigenvalues and eigenvectors of very large matrices. We typically obtain 100 eigenvalues of a $10\,000 \times 10\,000$ matrix in a few minutes on a regular workstation. The results presented here have been obtained by diagonalizing matrices of size up to 203 401 for a bandwidth equal to 903. For such matrices, we obtain 200 eigenvalues in 10 min on a Cray C98. The number of levels ($\simeq$36 000) that we are able to compute in this way is slightly larger than with usual boundary matching methods ($\simeq$20 000), which are nevertheless restricted to rational angles. On the other hand, very recent methods developed by Vergini *et al* [13] seems to be more efficient (they were able to reach an energy domain around the 142 000th state for the stadium billiard).

For the present study, various values of angle have been used:

$$\frac{3\pi}{10},\ \frac{(\sqrt{5}-1)\pi}{4},\ \frac{\pi}{\pi},\ \frac{\pi}{3},\ \frac{3\pi}{8} \text{ and } \frac{7\pi}{18} \tag{11}$$

for both $(++)$ and $(--)$ parity†. For all cases, only levels above the 10 000th one have been considered, to avoid peculiar effects in the statistics and at least 5000 levels (up to 24 000) have been used for each case. The convergence of the statistics has been checked by systematically varying the energy around which levels were taken. This is shown in figure 2, where we have plotted the following quantity:

$$\int_0^\infty \mathrm{d}s\ (N_0(s) - N_n(s))^2 \tag{12}$$

with respect to the number $n$, for $\frac{3\pi}{10}$ (top) and $\frac{\pi}{3}$ (bottom) billiards ($(++)$ parity). $N_0(s)$ is the cumulative NN distribution obtained with the 5000 highest states, whereas $N_n(s)$ is the cumulative NN distribution obtained with levels $n$ to $n + 4999$. One can thus clearly see that the statistics become energy independent (up to fluctuations) only for levels above the 10 000th state, which emphasizes the choice of keeping only those states.

In [7] it was shown that for the $\frac{3\pi}{10}(--)$ billiard, both NN and NNN statistics were following the formula (1). We, of course, reproduce this result, as shown in figure 3($a$). However, the same billiard, but with Neumann–Neumann boundary conditions, does not follow the same distribution laws but rather lies in between OE and SRDM distributions, as shown in figure 3($a$). The deviations are obviously much larger than statistical fluctuations. The difference is emphasized by looking at the behaviour of the NNN for small spacings (see figure 3($b$)). Indeed, whereas for the $(--)$ symmetry, the observed power law is $s^4$ in the cumulative distribution (i.e. $s^3$ for $P(1, s)$), it is close to $s^5$ for the $(++)$ case (i.e. $s^4$ for $P(1, s)$), which is the OE prediction. This dependency of the statistics with respect to the boundary conditions has already been observed in other systems like the 3D Anderson model [14]. However, the present results are more surprising as there are $\theta$ values for which there is practically no difference between the two symmetry classes. Indeed, figure 4 shows the NN (cumulative) distributions for $\frac{3\pi}{8}$ and $\frac{7\pi}{18}$. Besides the statistical fluctuations, one cannot distinguish between the two symmetry classes, whereas the distributions differ: $\frac{3\pi}{8}$ is well described by SRDM, whereas $\frac{7\pi}{18}$ lies between OE and SRDM.

† The $(--)$ parity for the $\frac{\pi}{3}$ rhombus is not shown as it is integrable.

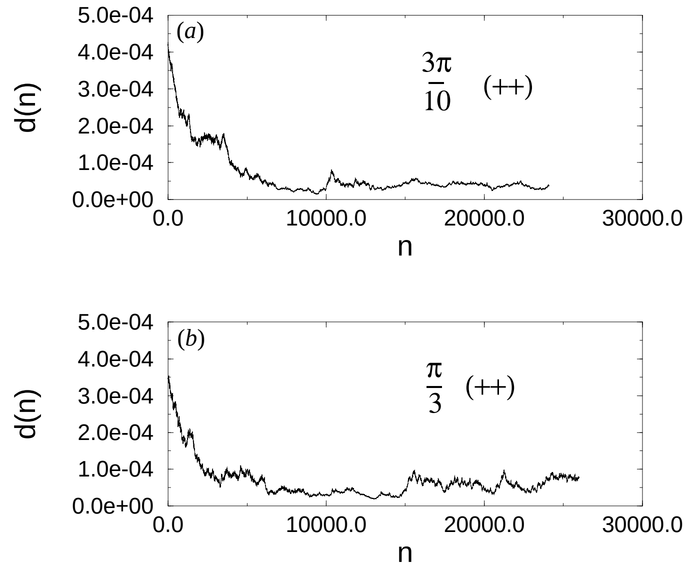


**Figure 2.** 'Difference' (see equation (12)) between the NN statistics obtained with the 5000 highest states and the NN statistics obtained with levels $n$ to $n+4999$, as a function of $n$, for both $\frac{3\pi}{10}$ (top) and $\frac{\pi}{3}$ (bottom) (++ parity). Above the 10 000th level, the distributions become energy independent (apart fluctuations).

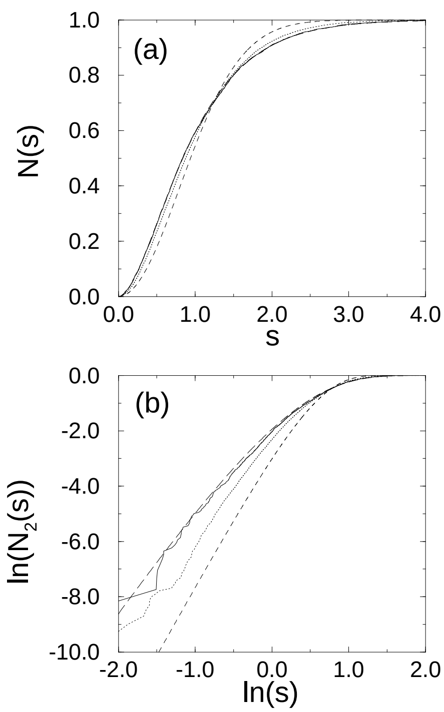


**Figure 3.** ($a$) The cumulative distribution of NN spacings for the $\frac{3\pi}{10}$ rhombus. The dotted curve corresponds to Neumann–Neumann (++) boundary conditions on both the $x$- and $y$-axis; the full curve corresponds to Dirichlet–Dirichlet (−−) boundary conditions. The two distributions are clearly different, the deviation being larger than statistical fluctuations. The (−−) symmetry class is exactly on the top of the distribution introduced by Bogomolny *et al* (SRDM) given by equation (1), corresponding to the long broken curve. The (++) distribution lies in between SRDM and OE prediction (given by the short broken curve). This difference is emphasized in ($b$) depicting the NNN distributions (cumulative) for the same billiards (ln–ln plot). Again, the (−−) (full curve) symmetry class is exactly on the top of SRDM (long broken curve), whereas the (++) symmetry class (dotted curve) lies in between SRDM and OE (short broken curve). In particular the behaviours for small spacing are very different: (−−) shows a $s^4$ power law, whereas it is $s^5$ for (++), the OE prediction.

The case of the $\frac{\pi}{3}$ billiard is the most peculiar, since the (−−) parity is integrable whereas the (++) spacing distributions agree with SRDM.

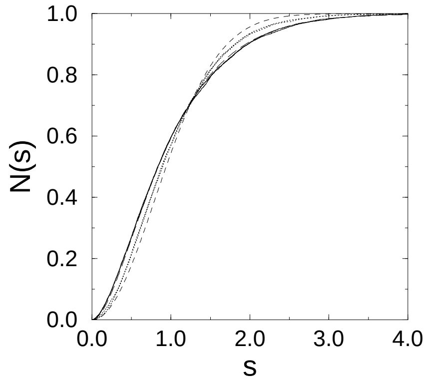


**Figure 4.** Spacing distributions (cumulative) for two rational billiards: $\frac{3\pi}{8}$ (full curves) and $\frac{7\pi}{18}$ (dotted curves), for both (++) and (−−) symmetry classes. In contrast to the $\frac{3\pi}{10}$ billiard (see figure 3), there is no difference between the two symmetry classes: for each billiard the two curves lie on top of each other. Furthermore, these two billiards show distinct spacing distributions, the $\frac{3\pi}{8}$ one corresponds exactly to SRDM (long broken curve) whereas the $\frac{7\pi}{18}$ one is much closer to OE prediction (short broken curve).

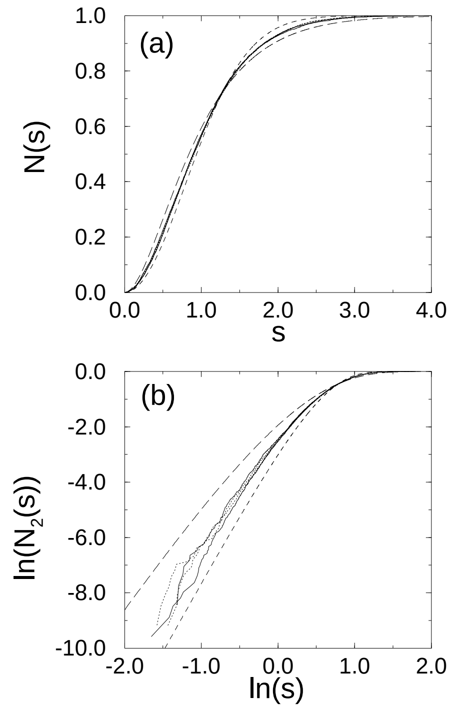


**Figure 5.** (*a*) NN and (*b*) NNN distributions for two irrational billiards: $\frac{\pi}{\pi}$ (full curves) and $\frac{(\sqrt{5}-1)\pi}{4}$ (dotted curves) for both (++) and (−−) symmetry classes. In contrast to the rational billiards, the genus of these billiard is 'infinite' (see text for explanation) and so the classical dynamics is ergodic. The fact that all the four distributions lie on top of each other is quite remarkable and may be related to the fact that these billiards have the 'same' genus. However, from the ergodicity one could expect the distributions to be OE-like, which is not the case. Rather, they lie between SRDM (long broken curve) and OE (short broken curve). Still, the small spacing behaviour of the NNN distributions shows a $s^5$ power law, i.e. corresponding to OE.

All the rhombi considered are not ergodic, as their genera are finite (e.g. two for the $\pi/3$-rhombus). In contrast, for an irrational angle the genus is 'infinite', and so one could expect a rather different behaviour. Although the concept of genus is applicable only to compact surfaces, we have stated the above phrase in quotes in the following sense: as an irrational rhombus is approximated via continued fraction expansion, the larger and larger denominators will appear, implying larger genus surfaces, until eventually 'infinite'. It is quite possible, and it may, in fact, be true, that this limit is singular. As a result, from the rational convergents, it may not be possible to say anything about the irrational billiard.

Figure 5 displays NN and NNN statistics for $\frac{\pi}{\pi}$ and $\frac{(\sqrt{5}-1)\pi}{4}$ (both symmetry classes) billiards. NN distributions are on top of each other, which is interesting if one believes that the genus is the relevant parameter. On the other hand, from the ergodicity, one could expect the distributions to be OE, which is not the case, even if the small spacing behaviour of NNN statistics seems to show the same power law $s^5$ (for cumulative). Thus, if [7] seems to give one class of universality, there must be other classes of universality lying between SRDM and OE, especially for irrational angles. The other possibility is that, although numerically stationary in a wide range of energy, the spacing distributions of the irrational rhombus may evolve exceptionally slowly to OE. If that is the case, one will probably have to find the final answer in a much higher energy range, for which other numerical methods will have to be used [13].

The present study also raises the question of the semiclassical understanding of the boundary dependence of the distributions. Due to a change in the boundary conditions, actions, Maslov indices and also the edge orbits will change resulting in a difference, but the whole explanation of this boundary dependence probably lies beyond these simple considerations. Spectral fluctuations in some of the pseudo-integrable billiards have been studied in detail using the periodic orbit theory. From the detailed information about the periodic orbits [4] it was shown that the spectral rigidity is non-universal [3] with a universal trend. We hope that the method presented here and the ensuing numerical results will help us to model the spectral fluctuations of these apparently simple non-integrable quantum systems.

To summarize: we have cast the problem of a particle in rhombus-shaped enclosure in a way that allows us to go to very high energies. This has led us to confidently obtain statistical results on spacing distributions which are well converged. Subsequently, we have shown that for some rational billiards, the fluctuations agree well with the results recently obtained [7]. However, we have given several examples where the recent model does not explain the obtained distributions. It has been shown that boundary conditions play an important role. Finally, for the irrational rhombus billiards, the distributions seem to be identical for the examples considered. Significantly though, the distribution is still not in the Wigner–Dyson family. We believe that these results point in the direction of having a family of universality classes which, in essence, leads to non-universality with a universal trend for pseudo-integrable billiards.

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